Analysis of Energy Harvesting from Vibrating Structure

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INTRODUCTION

In recent years energy harvesting has been an interesting topic for researchers due to reduction in power consumption of small electronics components. Piezoelectric material is used as a medium due to its properties which allow to change vibration motion into electrical energy and it is an ideal device because of its direct effect as used in sensor device. A cantilever beam with a harmonic force applied. Due to the stress in the piezoelectric material generate the electric current and voltage due to excitation of beam. The mathematical model has developed using Euler-Bernoulli beam theory and the analysis with the help of MATLAB. A power of 5.9 mW was generated at the resonance frequency 23.28 Hz.

Key Words: Energy Harvesting, Cantilever Beam, Piezoelectric Material, Vibration Frequency.

PIEZOELECTRIC CONSTITUTIVE EQUATION

The constitutive equations describing the piezoelectric property are based on the assumption that the total strain in the piezoelectric material is the sum of mechanical strain induced by mechanical stress and dielectric strain caused by the applied electric voltage. The electromechanical equations for a linear piezoelectric material can be written as

DIRECT PIEZOELECTRIC EFFECT

\[ \{D\} = \{e\} \cdot \{S\} + \{a\} \cdot \{E\} \]  

CONVERSE PIEZOELECTRIC EFFECT

\[ \{T\} = \{c_{ij}\} \cdot \{S\} - \{e\} \cdot \{E\} \]

Where \( \{D\} \) = electric displacement vector, \( \{T\} \) = stress vector, \( \{e\} \) = dielectric permittivity matrix, \( \{c_{ij}\} \) = elastic coefficient matrix, \( \{a\} \) = dielectric matrix, \( \{S\} \) = strain tensor \( \{E\} \) = electric field vector.

MATHEMATICAL MODELING

Fig. 1 shows the setup for the cantilever beam model. The PZT patch is attached to the beam near the clamped edge for maximum strain. For the estimated power that a PZT can produce from beam vibrations to be calculated, the moment that the PZT experiences must first be determined.
The Euler-Bernoulli method is used to model the cantilever beam. The governing undamped equation of motion for the beam for forced motion under zero initial conditions can be written as [11]:

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + E I \frac{\partial^4 w(x,t)}{\partial x^4} = F(t) \quad \ldots(3)$$

Where $w$ is the displacement of the beam, $\rho$ is the density of the steel beam, $A$ is the cross sectional area, and $F(t)$ is the external force applied to the beam. The boundary conditions are:

$$w(0, t) = 0$$
$$w_x(0, t) = 0$$
$$w_{xx}(L_p, t) = 0$$
$$w_{xxx}(L_p, t) = 0$$

Considering a harmonic forcing function applied to a single point on the beam, according to Fig. 1, we can write

$$\frac{\partial^2 w(x,t)}{\partial t^2} + C \frac{\partial^4 w(x,t)}{\partial x^4} = \frac{F}{\rho A} \sin(\omega t) \delta(x - L_p) \quad \ldots(4)$$

Where is the frequency, is the position of applied force from the clamped edge of the beam and $C = \frac{E I}{L_p \rho A}$.

The driving frequency will be equal to the beam’s first natural frequency because the largest deflections occur at the first natural frequency. The solution will take the form

$$w(x, t) = \sum_{i=1}^{N} q_i X_i(x) \quad \ldots(5)$$

Where $q_i$ is the $i$-th modal coordinate equation of the beam and $X_i(x)$ is the $i$-th mode shape of the beam. For consistency, only the first five mode shapes will be used. The general mode shape equation for a cantilever beam is

$$X_i(x) = \left( \cosh(\beta_i x) - \cos(\beta_i x) \right) - \frac{\sinh(h(\beta_i, L_p)) - \sin(\beta_i, L_p))}{\cosh(h(\beta_i, L_p)) + \cos(\beta_i, L_p))} \quad \ldots(6)$$

Where $L_p$ is the beam length, $\beta_i = \frac{\omega_i}{C^2}$ and $\omega_i$, the $i$-th natural frequency, is found from the characteristics equation

$$\cos(\beta_i, L_p) \cos h(\beta_i, L_p) = -1 \quad \ldots(7)$$

Using orthogonality, the external force can be simplified to the expression

$$F(t) = \frac{F}{\rho A} \sin(\omega t) X_i(L_p) \quad \ldots(8)$$

The convolution integral for any arbitrary input to evaluate $q_i$ is in the form:

$$q_i(t) = \frac{1}{\omega_i} e^{-i \omega_i t} \left[ F(t) e^{i \omega_i t} \sin(\omega_i (t - t')) dt' \right] \quad \ldots(9)$$

Where $\omega_d$ is the damped natural frequency and $\zeta$ is the damping ratio. The most common damping ratio values fall between 0.01 and 0.05. For simplicity, the damping ratio will be assumed to be the average of this range, 0.03, for the beam.

The next step is to calculate the beam curvature. The curvature of the beam can be estimated as

$$K(x, t) = \frac{\partial^2 w(x,t)}{\partial x^2} \quad \ldots(10)$$

To eliminate the dependence of length from the expression, the average curvature was evaluated as

$$\bar{K}(t) = \frac{1}{L_p} \int_{0}^{L_p} K(x,t) dx \quad \ldots(11)$$

Where the limits of integration are the lengths along the beam where the PZT starts and ends. Finally, the applied moment acting on the beam is

$$M(t) = E b I b \bar{K}(t) \quad \ldots(12)$$

**OUTPUT ELECTRICAL VOLTAGE AND POWER**

The Voltage on the PZT poling surfaces is related to the stress by [9]

$$V = g_{31} T_p \sigma_p \quad \ldots(13)$$

And voltage is evaluated as

$$V = \frac{-6 g_{31} M \varphi (1 + T)}{b l_p [1 + (\varphi T)^2 + 2 \varphi (2 + 3 T + 2 T^3)]} \quad \ldots(14)$$

Where $\varphi = \frac{E b E_p}{E_p l_p}$ and $T = \frac{l_p}{t_p}$.

Now the output power is evaluated as

$$P = \frac{V^2}{R} \quad \ldots(15)$$

Where $R$ is the resistance of the load.

**RESULTS AND DISCUSSIONS**

The parametric analysis was done by MATLAB. Table 1 shows the dimension and properties of beam and PZT patch used for this work:
Table 1: Dimensions and Properties of Beam and PZT

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam Thickness</td>
<td>0.025</td>
<td>m</td>
</tr>
<tr>
<td>Young’s Modulus</td>
<td>210×10^9</td>
<td>N/m</td>
</tr>
<tr>
<td>Density</td>
<td>7850</td>
<td>Kg/m</td>
</tr>
<tr>
<td>Length</td>
<td>0.10</td>
<td>m</td>
</tr>
<tr>
<td>Width</td>
<td>0.05</td>
<td>m</td>
</tr>
<tr>
<td>PZT Thickness</td>
<td>0.002</td>
<td>m</td>
</tr>
<tr>
<td>Young’s Modulus</td>
<td>62.5×10^9</td>
<td>N/m</td>
</tr>
<tr>
<td>Piezoelectric charge constant (d31)</td>
<td>-195×10^-12</td>
<td>C/N</td>
</tr>
<tr>
<td>Piezoelectric Voltage constant (g31)</td>
<td>-11×10^-3</td>
<td>Vm/N</td>
</tr>
</tbody>
</table>

In this paper, we concentrate to five natural frequencies which referring to five modes. Fig. 2 shows the mode shape of the beam. Table 2 shows the first five natural frequency of the cantilever beam.

Table 2: First five natural frequencies of cantilever beam

| Modes, i | Analytical Results |  |
|----------|--------------------|  |
|          | β/|ω₀ (rad/s) | f (Hz) |
| 1        | 1.8751 | 146.3 | 23.283 |
| 2        | 4.6941 | 916.8 | 145.912 |
| 3        | 7.8548 | 2567.1 | 408.559 |
| 4        | 10.9955 | 5030.4 | 800.613 |
| 5        | 14.1372 | 8315.6 | 1323.470 |

For the time being the external force’s magnitude, \( F_0 \), was set to 1N and it was applied at the free end of the beam. The PZT patch is attached to the beam near the clamped edge.

Fig. 3 shows the relationship between power and external load impedance which ranges from 0 to 1000KΩ with the interval of 1KΩ. There is optimal load impedance 359KΩ that gives the maximum output power of 1.023mW. This also follows the maximum power transfer theorem.

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Fig. 4 shows the power output as a function of PZT position. As expected, the highest power values are at position 0.05m of the beam length for the three different forces applied on the beam where the PZT is closest to the clamped end and experiences the largest strains.

The location of the piezoelectric material is important to the output power. Because the power produced by the PZT is directly proportional to the strain. It makes sense that the PZT must experience the largest strains that gives the maximum power output from the PZT patch. In this paper, the PZT length is 0.05m and the beam length is 0.30m and three different magnitude of force is applied on the beam. Fig. 4 shows the power output as a function of PZT position. As expected, the highest power values are at position 0.05m of the beam length for the three different forces applied on the beam where the PZT is closest to the clamped end and experiences the largest strains.

Fig. 5: Actual power as a function of PZT length
Fig. 5 shows the output power versus PZT length calculated by using current when the different magnitude force is applied at the free end of the beam. This gives the optimal length = 0.075 m for the 0.3 m length of the beam i.e. to get the maximum power the length of the PZT patch should be one fourth of the beam. Next the thickness ratio of the beam and PZT patch also plays an important role in the power generation. Fig. 6 shows the optimum thickness ratio of beam and PZT patch should be 0.21 at which the maximum output power occurs.

![Graph showing power versus thickness ratio of beam and PZT](image)

The power obtained from the vibrating structure using piezoelectric material also depends upon the frequency of applied force. Fig. 7 shows the variation of power with the frequency at the three fixed resistance of (R = 10kΩ, 350kΩ, and 500kΩ).

![Graph showing variation of power with frequency](image)

From Fig. 7 it clears the maximum output power obtained at the first natural frequency i.e. 23.283 Hz for three fixed resistance. The amplitude of output power decreases with increase in the frequency.

At the optimum parameter the maximum power obtained from the beam model is the 5.9mW at the external load impedance of 350kΩ when the 1N of force is applied. The power generated is 476% more that the power generated without the optimum variable.

**CONCLUSIONS**

This paper presents a parametric analysis of energy harvesting for cantilever beam model based on Euler-Bernoulli beam theory. The harvester is assumed to be excited due to the harmonic force applied at the beam in the transverse direction. From this analysis it clears that to get the maximum power the beam should be excited at the first natural frequency and the external impedance should be equal to the internal resistance. The PZT patch should be attached near the clamped edge, the thickness of the beam should be 21% of the PZT patch and the length of the patch should be one fourth of the beam length to get the optimum power.

**REFERENCES**

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